The Mean and Variance Of The Product Of Two Lognormally-Distributed Random Variates

Gary Schurman MBE, CFA

September, 2012

In this white paper we will calculate the mean and variance of the product of two lognormally-distributed random variates. The equation for the product of these two random variates is...

$$A\,e^{x}\,B\,e^{y} \tag{1}$$

In Equation (1) above A and B are constants and x and y are normally-distributed random variates that may or may not be correlated. Note that taking the exponential of a normally-distributed random variate results in a lognormally-distributed random variate.

Our Hypothetical Problem

We are tasked with calculating the mean and variance of two lognormally-distributed random variates. The assumptions that we will use in our problem are as follows...

Symbol	Description	Value
A	Constant multiplier	100.00
B	Constant multipier	0.50
μ_x	Mean of x	0.20
μ_y	Mean of y	0.05
σ_x	Std deviation of x	0.40
σ_y	Std deviation of y	0.10
$\rho_{x,y}$	Correlation of x and y	0.60

Question: What is the mean and variance of the product equation $A\,e^{x}\,B\,e^{y}$?

Setting Up The Problem

The distribution (mean and variance) of the random variate x is...

$$x \sim N\left[\mu_x, \sigma_x^2\right] \tag{2}$$

After normalizing the random variate x in Equation (2) above the equation for x becomes...

$$x = \mu_x + \sigma_x\,z_1 \text{ ...where... } z_1 \sim N\left[0, 1\right] \tag{3}$$

The distribution (mean and variance) of the random variate y is...

$$y \sim N\left[\mu_y, \sigma_y^2\right] \tag{4}$$

After normalizing the random variate y in Equation (4) and adding dependence (i.e. correlation) on the random variate z_1 in Equation (3) the equation for y becomes...

$$y = \mu_y + \rho_{x,y}\,\sigma_y\,z_1 + \sqrt{1-\rho_{x,y}^2}\,\sigma_y\,z_2 \text{ ...where... } z_2 \sim N\left[0, 1\right] \tag{5}$$

Note that the process of normalization does not affect correlation and therefore...

$$\rho_{x,y} = \rho_{z_1,z_2} \tag{6}$$

We will need an equation for the product of our two lognormally-distributed random variates. Using Equations (3) and (5) above this equation is...

$$\begin{aligned} A\,e^x B\,e^y &= A\,Exp\left\{\mu_x + \sigma_x\,z_1\right\} B\,Exp\left\{\mu_y + \rho_{x,y}\,\sigma_y\,z_1 + \sqrt{1-\rho_{x,y}^2}\,\sigma_y\,z_2\right\} \\ &= A\,B\,Exp\left\{\mu_x + \sigma_x\,z_1 + \mu_y + \rho_{x,y}\,\sigma_y\,z_1 + \sqrt{1-\rho_{x,y}^2}\,\sigma_y\,z_2\right\} \\ &= A\,B\,Exp\left\{\mu_x + \mu_y + \left(\sigma_x + \rho_{x,y}\,\sigma_y\right) z_1 + \sqrt{1-\rho_{x,y}^2}\,\sigma_y\,z_2\right\} \end{aligned} \tag{7}$$

We will also need an equation for the square of the product of our two lognormally-distributed random variates. Using Equations (3) and (5) above this equation is...

$$\begin{aligned} \left(A\,e^x B\,e^y\right)^2 &= A^2 e^{2x} B^2\,e^{2y} \\ &= A^2\,Exp\left\{2\mu_x + 2\sigma_x\,z_1\right\} B^2\,Exp\left\{2\mu_y + 2\rho_{x,y}\,\sigma_y\,z_1 + 2\sqrt{1-\rho_{x,y}^2}\,\sigma_y\,z_2\right\} \\ &= A^2\,B^2\,Exp\left\{2\mu_x + 2\sigma_x\,z_1 + 2\mu_y + 2\rho_{x,y}\,\sigma_y\,z_1 + 2\sqrt{1-\rho_{x,y}^2}\,\sigma_y\,z_2\right\} \\ &= A^2\,B^2\,Exp\left\{2\mu_x + 2\mu_y + 2\left(\sigma_x + \rho_{x,y}\,\sigma_y\right) z_1 + 2\sqrt{1-\rho_{x,y}^2}\,\sigma_y\,z_2\right\} \end{aligned} \tag{8}$$

To make Equations (7) and (8) easier to work with we will make the following simplifying definitions...

$$\theta_1 = \sigma_x + \rho_{x,y}\,\sigma_y \tag{9}$$

$$\theta_2 = \sqrt{1-\rho_{x,y}^2}\,\sigma_y \tag{10}$$

We will also need the square of θ_1, the square of θ_2 and the sum of the squares which are...

$$\theta_1^2 = \sigma_x^2 + \rho_{x,y}^2\sigma_y^2 + 2\rho_{x,y}\sigma_x\sigma_y \tag{11}$$

$$\theta_2^2 = \sigma_y^2 - \rho_{x,y}^2\sigma_y^2 \tag{12}$$

$$\theta_1^2 + \theta_2^2 = \sigma_x^2 + \sigma_y^2 + 2\rho_{x,y}\sigma_x\sigma_y \tag{13}$$

Using the definitions in Equations (9) and (10) above we can rewrite Equation (7) as...

$$A\,e^x B\,e^y = A\,B\,Exp\left\{\mu_x + \mu_y + \theta_1\,z_1 + \theta_2\,z_2\right\} = A\,B\,e^{\mu_x+\mu_y}\,e^{\theta_1\,z_1}\,e^{\theta_2\,z_2} \tag{14}$$

Using the definitions in Equations (9) and (10) above we can rewrite Equation (8) as...

$$\left(A\,e^x B\,e^y\right)^2 = A^2\,B^2\,Exp\left\{2\,\mu_x + 2\,\mu_y + 2\,\theta_1\,z_1 + 2\,\theta_2\,z_2\right\} = A^2\,B^2\,e^{2\,\mu_x+2\,\mu_y}\,e^{2\,\theta_1\,z_1}\,e^{2\,\theta_2\,z_2} \tag{15}$$

The First Moment Of The Distribution

Using Equation (14) above the equation for the first moment of the distribution of the product of two lognormally-distributed random variates is...

$$\begin{aligned} \mathbb{E}\left[A\,e^x B\,e^y\right] &= \int_{z_1=-\infty}^{z_1=\infty}\int_{z_2=-\infty}^{z_2=\infty} A\,B\,e^{\mu_x+\mu_y}\,\frac{1}{\sqrt{2\,\pi}}\,e^{-\frac{1}{2}z_1^2}\,e^{\theta_1\,z_1}\,\frac{1}{\sqrt{2\,\pi}}\,e^{-\frac{1}{2}z_2^2}\,e^{\theta_2\,z_2}\,\delta z_2\,\delta z_1 \\ &= A\,B\,e^{\mu_x+\mu_y}\int_{z_1=-\infty}^{z_1=\infty}\int_{z_2=-\infty}^{z_2=\infty}\frac{1}{\sqrt{2\,\pi}}\,e^{-\frac{1}{2}z_1^2}\,e^{\theta_1\,z_1}\,\frac{1}{\sqrt{2\,\pi}}\,e^{-\frac{1}{2}z_2^2}\,e^{\theta_2\,z_2}\,\delta z_2\,\delta z_1 \end{aligned} \tag{16}$$

To solve the expectation in Equation (16) we first evaluate the inside integral which is...

$$\begin{aligned}\int_{z_2=-\infty}^{z_2=\infty} \frac{1}{\sqrt{2\,\pi}}\,e^{-\frac{1}{2}z_1^2}\,e^{\theta_1\,z_1}\,\frac{1}{\sqrt{2\,\pi}}\,e^{-\frac{1}{2}z_2^2}\,e^{\theta_2\,z_2}\,\delta z_2 &= \frac{1}{\sqrt{2\,\pi}}e^{-\frac{1}{2}z_1^2}e^{\theta_1\,z_1}\int_{z_2=-\infty}^{z_2=\infty}\frac{1}{\sqrt{2\,\pi}}\,e^{-\frac{1}{2}z_2^2}\,e^{\theta_2\,z_2}\,\delta z_2\\ &= \frac{1}{\sqrt{2\,\pi}}e^{-\frac{1}{2}z_1^2}e^{\theta_1\,z_1}\int_{z_2=-\infty}^{z_2=\infty}\frac{1}{\sqrt{2\,\pi}}\,e^{-\frac{1}{2}z_2^2+\theta_2\,z_2}\,\delta z_2\\ &= \frac{1}{\sqrt{2\,\pi}}e^{-\frac{1}{2}z_1^2}e^{\theta_1\,z_1}\int_{z_2=-\infty}^{z_2=\infty}\frac{1}{\sqrt{2\,\pi}}\,e^{-\frac{1}{2}z_2^2+\theta_2\,z_2+\frac{1}{2}\theta_2^2-\frac{1}{2}\theta_2^2}\,\delta z_2\\ &= \frac{1}{\sqrt{2\,\pi}}e^{-\frac{1}{2}z_1^2}e^{\theta_1\,z_1}\int_{z_2=-\infty}^{z_2=\infty}\frac{1}{\sqrt{2\,\pi}}\,e^{-\frac{1}{2}z_2^2+\theta_2\,z_2-\frac{1}{2}\theta_2^2}\,e^{\frac{1}{2}\theta_2^2}\,\delta z_2\\ &= \frac{1}{\sqrt{2\,\pi}}e^{-\frac{1}{2}z_1^2}e^{\theta_1\,z_1}\,e^{\frac{1}{2}\theta_2^2}\int_{z_2=-\infty}^{z_2=\infty}\frac{1}{\sqrt{2\,\pi}}\,e^{-\frac{1}{2}z_2^2+\theta_2\,z_2-\frac{1}{2}\theta_2^2}\,\delta z_2\end{aligned} \tag{17}$$

Noting that...

$$-\frac{1}{2}\bigg(z_2-\theta_2\bigg)^2 = -\frac{1}{2}z_2^2+\theta_2\,z_2-\frac{1}{2}\theta_2^2 \tag{18}$$

We can rewrite Equation (17) as...

$$\int_{z_2=-\infty}^{z_2=\infty} \frac{1}{\sqrt{2\,\pi}}\,e^{-\frac{1}{2}z_1^2}\,e^{\theta_1\,z_1}\,\frac{1}{\sqrt{2\,\pi}}\,e^{-\frac{1}{2}z_2^2}\,e^{\theta_2\,z_2}\,\delta z_2 = \frac{1}{\sqrt{2\,\pi}}e^{-\frac{1}{2}z_1^2}e^{\theta_1\,z_1}\,e^{\frac{1}{2}\theta_2^2}\int_{z_2=-\infty}^{z_2=\infty}\frac{1}{\sqrt{2\,\pi}}\,e^{-\frac{1}{2}(z_2-\theta_2)^2}\,\delta z_2 \tag{19}$$

After making the following definition...

$$\phi_2 = z_2-\theta_2\ \text{ ...where... }\ \frac{\delta\phi_2}{\delta z_2}=1\ \text{ ...such that... }\ \delta z_2=\delta\phi_2 \tag{20}$$

We can rewrite Equation (19) as...

$$\begin{aligned}\int_{z_2=-\infty}^{z_2=\infty} \frac{1}{\sqrt{2\,\pi}}\,e^{-\frac{1}{2}z_1^2}\,e^{\theta_1\,z_1}\,\frac{1}{\sqrt{2\,\pi}}\,e^{-\frac{1}{2}z_2^2}\,e^{\theta_2\,z_2}\,\delta z_2 &= \frac{1}{\sqrt{2\,\pi}}e^{-\frac{1}{2}z_1^2}e^{\theta_1\,z_1}\,e^{\frac{1}{2}\theta_2^2}\int_{\phi_2=-\infty-\theta_2}^{\phi_2=\infty-\theta_2}\frac{1}{\sqrt{2\,\pi}}\,e^{-\frac{1}{2}\phi_2^2}\,\delta\phi_2\\ &= \frac{1}{\sqrt{2\,\pi}}e^{-\frac{1}{2}z_1^2}e^{\theta_1\,z_1}\,e^{\frac{1}{2}\theta_2^2}\int_{\phi_2=-\infty}^{\phi_2=\infty}\frac{1}{\sqrt{2\,\pi}}\,e^{-\frac{1}{2}\phi_2^2}\,\delta\phi_2\\ &= \frac{1}{\sqrt{2\,\pi}}e^{-\frac{1}{2}z_1^2}e^{\theta_1\,z_1}\,e^{\frac{1}{2}\theta_2^2}\end{aligned} \tag{21}$$

After replacing the inside integral of Equation (16) with Equation (21) we can rewrite Equation (16) as...

$$\begin{aligned}\mathbb{E}\bigg[A\,e^x\,B\,e^y\bigg] &= A\,B\,e^{\mu_x+\mu_y}\int_{z_1=-\infty}^{z_1=\infty}\frac{1}{\sqrt{2\,\pi}}e^{-\frac{1}{2}z_1^2}e^{\theta_1\,z_1}\,e^{\frac{1}{2}\theta_2^2}\,\delta z_1\\ &= A\,B\,e^{\mu_x+\mu_y}\,e^{\frac{1}{2}\theta_2^2}\int_{z_1=-\infty}^{z_1=\infty}\frac{1}{\sqrt{2\,\pi}}e^{-\frac{1}{2}z_1^2}e^{\theta_1\,z_1}\,\delta z_1\end{aligned} \tag{22}$$

We then evaluate the remaining integral in Equation (22) which is...

$$\begin{aligned}\int_{z_1=-\infty}^{z_1=\infty}\frac{1}{\sqrt{2\,\pi}}e^{-\frac{1}{2}z_1^2}e^{\theta_1\,z_1}\,\delta z_1 &= \int_{z_1=-\infty}^{z_1=\infty}\frac{1}{\sqrt{2\,\pi}}e^{-\frac{1}{2}z_1^2+\theta_1\,z_1}\,\delta z_1\\ &= \int_{z_1=-\infty}^{z_1=\infty}\frac{1}{\sqrt{2\,\pi}}e^{-\frac{1}{2}z_1^2+\theta_1\,z_1+\frac{1}{2}\theta_1^2-\frac{1}{2}\theta_1^2}\,\delta z_1\\ &= \int_{z_1=-\infty}^{z_1=\infty}\frac{1}{\sqrt{2\,\pi}}e^{-\frac{1}{2}z_1^2+\theta_1\,z_1-\frac{1}{2}\theta_1^2}\,e^{\frac{1}{2}\theta_1^2}\,\delta z_1\\ &= e^{\frac{1}{2}\theta_1^2}\int_{z_1=-\infty}^{z_1=\infty}\frac{1}{\sqrt{2\,\pi}}e^{-\frac{1}{2}z_1^2+\theta_1\,z_1-\frac{1}{2}\theta_1^2}\,\delta z_1\end{aligned} \tag{23}$$

Noting that...

$$-\frac{1}{2}\bigg(z_1-\theta_1\bigg)^2 = -\frac{1}{2}z_1^2+\theta_1\,z_1-\frac{1}{2}\theta_1^2 \tag{24}$$

We can rewrite Equation (23) as...

$$\int_{z_1=-\infty}^{z_1=\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z_1^2} e^{\theta_1 z_1} \, \delta z_1 = e^{\frac{1}{2}\theta_1^2} \int_{z_1=-\infty}^{z_1=\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(z_1-\theta_1)^2} \, \delta z_1 \tag{25}$$

After making the following definition...

$$\phi_1 = z_1 - \theta_1 \;\; \text{...where...} \;\; \frac{\delta \phi_1}{\delta z_1} = 1 \;\; \text{...such that...} \;\; \delta z_1 = \delta \phi_1 \tag{26}$$

We can rewrite Equation (25) as...

$$\begin{aligned} \int_{z_1=-\infty}^{z_1=\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z_1^2} e^{\theta_1 z_1} \, \delta z_1 &= e^{\frac{1}{2}\theta_1^2} \int_{\phi_1=-\infty-\theta_1}^{\phi_1=\infty-\theta_1} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}\phi_1^2} \, \delta \phi_1 \\ &= e^{\frac{1}{2}\theta_1^2} \int_{\phi_1=-\infty}^{\phi_1=\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}\phi_1^2} \, \delta \phi_1 \\ &= e^{\frac{1}{2}\theta_1^2} \end{aligned} \tag{27}$$

After replacing the integral of Equation (22) with Equation (27) we can rewrite Equation (22) as...

$$\begin{aligned} \mathbb{E}\Bigl[A\, e^x B\, e^y\Bigr] &= A\, B\, e^{\mu_x+\mu_y} \, e^{\frac{1}{2}\theta_2^2} \, e^{\frac{1}{2}\theta_1^2} \\ &= A\, B\, e^{\mu_x+\mu_y+\frac{1}{2}(\theta_2^2+\theta_1^2)} \end{aligned} \tag{28}$$

Using Equation (13) above we can rewrite Equation (28) as...

$$\mathbb{E}\Bigl[A\, e^x B\, e^y\Bigr] = A\, B\, e^{\mu_x+\mu_y+\frac{1}{2}(\sigma_x^2+\sigma_y^2+2\rho_{x,y}\sigma_x\sigma_y)} \tag{29}$$

Done! Equation (29) above is the equation for the first moment of the distribution of the product of two lognormally-distributed random variates.

The Second Moment Of The Distribution

Using Equation (15) above the equation for the second moment of the distribution of the product of two lognormally-distributed random variates as per Equation (1) above is...

$$\begin{aligned} \mathbb{E}\Biggl[\Bigl(A\, e^x B\, e^y\Bigr)^2\Biggr] &= \int_{z_1=-\infty}^{z_1=\infty} \int_{z_2=-\infty}^{z_2=\infty} A^2\, B^2\, e^{2\,\mu_x+2\,\mu_y} \, \frac{1}{\sqrt{2\pi}} \, e^{-\frac{1}{2}z_1^2} \, e^{2\,\theta_1\, z_1} \, \frac{1}{\sqrt{2\pi}} \, e^{-\frac{1}{2}z_2^2} \, e^{2\,\theta_2\, z_2} \, \delta z_2 \, \delta z_1 \\ &= A^2\, B^2\, e^{2\,\mu_x+2\,\mu_y} \int_{z_1=-\infty}^{z_1=\infty} \int_{z_2=-\infty}^{z_2=\infty} \frac{1}{\sqrt{2\pi}} \, e^{-\frac{1}{2}z_1^2} \, e^{2\,\theta_1\, z_1} \, \frac{1}{\sqrt{2\pi}} \, e^{-\frac{1}{2}z_2^2} \, e^{2\,\theta_2\, z_2} \, \delta z_2 \, \delta z_1 \end{aligned} \tag{30}$$

To solve the expectation in Equation (30) we first evaluate the inside integral which is...

$$\begin{aligned} \int_{z_2=-\infty}^{z_2=\infty} \frac{1}{\sqrt{2\pi}} \, e^{-\frac{1}{2}z_1^2} \, e^{2\,\theta_1\, z_1} \, \frac{1}{\sqrt{2\pi}} \, e^{-\frac{1}{2}z_2^2} \, e^{2\,\theta_2\, z_2} \, \delta z_2 &= \frac{1}{\sqrt{2\pi}} \, e^{-\frac{1}{2}z_1^2} \, e^{2\,\theta_1\, z_1} \int_{z_2=-\infty}^{z_2=\infty} \frac{1}{\sqrt{2\pi}} \, e^{-\frac{1}{2}z_2^2} \, e^{2\,\theta_2\, z_2} \, \delta z_2 \\ &= \frac{1}{\sqrt{2\pi}} \, e^{-\frac{1}{2}z_1^2} \, e^{2\,\theta_1\, z_1} \int_{z_2=-\infty}^{z_2=\infty} \frac{1}{\sqrt{2\pi}} \, e^{-\frac{1}{2}z_2^2+2\,\theta_2\, z_2} \, \delta z_2 \\ &= \frac{1}{\sqrt{2\pi}} \, e^{-\frac{1}{2}z_1^2} \, e^{2\,\theta_1\, z_1} \int_{z_2=-\infty}^{z_2=\infty} \frac{1}{\sqrt{2\pi}} \, e^{-\frac{1}{2}z_2^2+2\,\theta_2\, z_2+2\,\theta_2^2-2\,\theta_2^2} \, \delta z_2 \\ &= \frac{1}{\sqrt{2\pi}} \, e^{-\frac{1}{2}z_1^2} \, e^{2\,\theta_1\, z_1} \int_{z_2=-\infty}^{z_2=\infty} \frac{1}{\sqrt{2\pi}} \, e^{-\frac{1}{2}z_2^2+2\,\theta_2\, z_2-2\,\theta_2^2} \, e^{2\,\theta_2^2} \, \delta z_2 \\ &= \frac{1}{\sqrt{2\pi}} \, e^{-\frac{1}{2}z_1^2} \, e^{2\,\theta_1\, z_1} \, e^{2\,\theta_2^2} \int_{z_2=-\infty}^{z_2=\infty} \frac{1}{\sqrt{2\pi}} \, e^{-\frac{1}{2}z_2^2+2\,\theta_2\, z_2-2\,\theta_2^2} \, \delta z_2 \end{aligned} \tag{31}$$

Noting that...

$$-\frac{1}{2}\left(z_2 - 2\,\theta_2\right)^2 = -\frac{1}{2}z_2^2 + 2\,\theta_2\,z_2 - 2\,\theta_2^2 \tag{32}$$

We can rewrite Equation (31) as...

$$\int_{z_2=-\infty}^{z_2=\infty} \frac{1}{\sqrt{2\,\pi}}\,e^{-\frac{1}{2}z_1^2}\,e^{2\,\theta_1\,z_1}\,\frac{1}{\sqrt{2\,\pi}}\,e^{-\frac{1}{2}z_2^2}\,e^{2\,\theta_2\,z_2}\,\delta z_2 = \frac{1}{\sqrt{2\,\pi}}\,e^{-\frac{1}{2}z_1^2}\,e^{2\,\theta_1\,z_1}\,e^{2\,\theta_2^2}\int_{z_2=-\infty}^{z_2=\infty}\frac{1}{\sqrt{2\,\pi}}\,e^{-\frac{1}{2}(z_2-2\,\theta_2)^2}\,\delta z_2 \tag{33}$$

After making the following definition...

$$\omega_2 = z_2 - 2\,\theta_2 \text{ ...where... } \frac{\delta\omega_2}{\delta z_2} = 1 \text{ ...such that... } \delta z_2 = \delta\omega_2 \tag{34}$$

We can rewrite Equation (33) as...

$$\begin{aligned}\int_{z_2=-\infty}^{z_2=\infty} \frac{1}{\sqrt{2\,\pi}}\,e^{-\frac{1}{2}z_1^2}\,e^{2\,\theta_1\,z_1}\,\frac{1}{\sqrt{2\,\pi}}\,e^{-\frac{1}{2}z_2^2}\,e^{2\,\theta_2\,z_2}\,\delta z_2 &= \frac{1}{\sqrt{2\,\pi}}\,e^{-\frac{1}{2}z_1^2}\,e^{2\,\theta_1\,z_1}\,e^{2\,\theta_2^2}\int_{\omega_2=-\infty-2\,\theta_2}^{\omega_2=\infty-2\,\theta_2}\frac{1}{\sqrt{2\,\pi}}\,e^{-\frac{1}{2}\omega_2^2}\,\delta\omega_2\\ &= \frac{1}{\sqrt{2\,\pi}}\,e^{-\frac{1}{2}z_1^2}\,e^{2\,\theta_1\,z_1}\,e^{2\,\theta_2^2}\int_{\omega_2=-\infty}^{\omega_2=\infty}\frac{1}{\sqrt{2\,\pi}}\,e^{-\frac{1}{2}\omega_2^2}\,\delta\omega_2\\ &= \frac{1}{\sqrt{2\,\pi}}\,e^{-\frac{1}{2}z_1^2}\,e^{2\,\theta_1\,z_1}\,e^{2\,\theta_2^2}\end{aligned} \tag{35}$$

After replacing the inside integral of Equation (30) with Equation (35) we can rewrite Equation (30) as...

$$\begin{aligned}\mathbb{E}\left[\left(A\,e^x\,B\,e^y\right)^2\right] &= A^2\,B^2\,e^{2\,\mu_x+2\,\mu_y}\int_{z_1=-\infty}^{z_1=\infty}\frac{1}{\sqrt{2\,\pi}}\,e^{-\frac{1}{2}z_1^2}\,e^{2\,\theta_1\,z_1}\,e^{2\,\theta_2^2}\,\delta z_1\\ &= A^2\,B^2\,e^{2\,\mu_x+2\,\mu_y}\,e^{2\,\theta_2^2}\int_{z_1=-\infty}^{z_1=\infty}\frac{1}{\sqrt{2\,\pi}}\,e^{-\frac{1}{2}z_1^2}\,e^{2\,\theta_1\,z_1}\,\delta z_1\end{aligned} \tag{36}$$

We then evaluate the remaining integral in Equation (36) which is...

$$\begin{aligned}\int_{z_1=-\infty}^{z_1=\infty}\frac{1}{\sqrt{2\,\pi}}\,e^{-\frac{1}{2}z_1^2}\,e^{2\,\theta_1\,z_1}\,\delta z_1 &= \int_{z_1=-\infty}^{z_1=\infty}\frac{1}{\sqrt{2\,\pi}}\,e^{-\frac{1}{2}z_1^2+2\,\theta_1\,z_1}\,\delta z_1\\ &= \int_{z_1=-\infty}^{z_1=\infty}\frac{1}{\sqrt{2\,\pi}}\,e^{-\frac{1}{2}z_1^2+2\,\theta_1\,z_1+2\,\theta_1^2-2\,\theta_1^2}\,\delta z_1\\ &= \int_{z_1=-\infty}^{z_1=\infty}\frac{1}{\sqrt{2\,\pi}}\,e^{-\frac{1}{2}z_1^2+2\,\theta_1\,z_1-2\,\theta_1^2}\,e^{2\,\theta_1^2}\,\delta z_1\\ &= e^{2\,\theta_1^2}\int_{z_1=-\infty}^{z_1=\infty}\frac{1}{\sqrt{2\,\pi}}\,e^{-\frac{1}{2}z_1^2+2\,\theta_1\,z_1-2\,\theta_1^2}\,\delta z_1\end{aligned} \tag{37}$$

Noting that...

$$-\frac{1}{2}\left(z_1 - 2\,\theta_1\right)^2 = -\frac{1}{2}z_1^2 + 2\,\theta_1\,z_1 - 2\,\theta_1^2 \tag{38}$$

We can rewrite Equation (37) as...

$$\int_{z_1=-\infty}^{z_1=\infty}\frac{1}{\sqrt{2\,\pi}}\,e^{-\frac{1}{2}z_1^2}\,e^{2\,\theta_1\,z_1}\,\delta z_1 = e^{2\,\theta_1^2}\int_{z_1=-\infty}^{z_1=\infty}\frac{1}{\sqrt{2\,\pi}}\,e^{-\frac{1}{2}(z_1-2\,\theta_1)^2}\,\delta z_1 \tag{39}$$

After making the following definition...

$$\omega_1 = z_1 - 2\,\theta_1 \text{ ...where... } \frac{\delta\omega_1}{\delta z_1} = 1 \text{ ...such that... } \delta z_1 = \delta\omega_1 \tag{40}$$

We can rewrite Equation (39) as...

$$\begin{aligned}\int_{z_1=-\infty}^{z_1=\infty}\frac{1}{\sqrt{2\,\pi}}\,e^{-\frac{1}{2}z_1^2}\,e^{2\,\theta_1\,z_1}\,\delta z_1 &= e^{2\,\theta_1^2}\int_{\omega_1=-\infty-2\,\theta_1}^{\omega_1=\infty-2\,\theta_1}\frac{1}{\sqrt{2\,\pi}}\,e^{-\frac{1}{2}\omega_1^2}\,\delta\omega_1\\ &= e^{2\,\theta_1^2}\int_{\omega_1=-\infty}^{\omega_1=\infty}\frac{1}{\sqrt{2\,\pi}}\,e^{-\frac{1}{2}\omega_1^2}\,\delta\omega_1\\ &= e^{2\,\theta_1^2}\end{aligned} \tag{41}$$

After replacing the integral of Equation (36) with Equation (41) we can rewrite Equation (36) as...

$$\mathbb{E}\Bigg[\bigg(A\,e^x B\,e^y\bigg)^2\Bigg] = A^2\,B^2\,e^{\,2\,\mu_x+2\,\mu_y}\,e^{2\,\theta_2^2}\,e^{2\,\theta_1^2}$$
$$= A^2\,B^2\,e^{\,2\,\mu_x+2\,\mu_y+2\,(\theta_2^2+\theta_1^2)} \tag{42}$$

Using Equation (13) above we can rewrite Equation (42) as...

$$\mathbb{E}\Bigg[\bigg(A\,e^x B\,e^y\bigg)^2\Bigg] = A^2\,B^2\,e^{\,2\,\mu_x+2\,\mu_y+2\,(\sigma_x^2+\sigma_y^2+2\rho_{x,y}\sigma_x\sigma_y)} \tag{43}$$

Done! Equation (43) above is the equation for the second moment of the distribution of the product of two lognormally-distributed random variates.

The Mean And Variance Of The Distribution

The mean of the distribution is the first moment of the distribution. Per Equation (29) above the mean of the distribution of the product of two lognormally-distributed random variates is...

$$mean = \mathbb{E}\Bigg[A\,e^x B\,e^y\Bigg] = A\,B\,e^{\,\mu_x+\mu_y+\frac{1}{2}(\sigma_x^2+\sigma_y^2+2\rho_{x,y}\sigma_x\sigma_y)} \tag{44}$$

The variance of the distribution is the second moment of the distribution minus the square of the first moment of the distribution. Using Equations (29) and (43) above the variance of the distribution of the product of two lognormally-distributed random variates is...

$$variance = \mathbb{E}\Bigg[\bigg(A\,e^x B\,e^y\bigg)^2\Bigg] - \Bigg(\mathbb{E}\Bigg[A\,e^x B\,e^y\Bigg]\Bigg)^2$$
$$= A^2\,B^2\,e^{\,2\,\mu_x+2\,\mu_y+2\,(\sigma_x^2+\sigma_y^2+2\rho_{x,y}\sigma_x\sigma_y)} - \Bigg(A\,B\,e^{\,\mu_x+\mu_y+\frac{1}{2}(\sigma_x^2+\sigma_y^2+2\rho_{x,y}\sigma_x\sigma_y)}\Bigg)^2$$
$$= A^2\,B^2\,e^{\,2\,\mu_x+2\,\mu_y+2\,(\sigma_x^2+\sigma_y^2+2\rho_{x,y}\sigma_x\sigma_y)} - A^2\,B^2\,e^{\,2\,\mu_x+2\mu_y+\sigma_x^2+\sigma_y^2+2\rho_{x,y}\sigma_x\sigma_y}$$
$$= A^2\,B^2\,\Bigg(e^{\,2\,\mu_x+2\,\mu_y+2\,(\sigma_x^2+\sigma_y^2+2\rho_{x,y}\sigma_x\sigma_y)} - e^{\,2\,\mu_x+2\mu_y+\sigma_x^2+\sigma_y^2+2\rho_{x,y}\sigma_x\sigma_y}\Bigg)$$
$$= A^2\,B^2\,e^{\,2\,\mu_x+2\,\mu_y}\,\Bigg(e^{\,2\,(\sigma_x^2+\sigma_y^2+2\rho_{x,y}\sigma_x\sigma_y)} - e^{\,\sigma_x^2+\sigma_y^2+2\rho_{x,y}\sigma_x\sigma_y}\Bigg) \tag{45}$$

The Solution To Our Problem

Using Equation (44) above the mean of the distribution is...

$$mean = A\,B\,e^{\,\mu_x+\mu_y+\frac{1}{2}(\sigma_x^2+\sigma_y^2+2\rho_{x,y}\sigma_x\sigma_y)}$$
$$= (100)(0.50)\,Exp\Bigg\{0.20+0.05+\frac{1}{2}\Bigg(0.40^2+0.10^2+(2)(0.60)(0.40)(0.10)\Bigg)\Bigg\}$$
$$= 71.59 \tag{46}$$

Using Equation (45) above the variance of the distribution is...

$$variance = A^2\,B^2\,e^{\,2\,\mu_x+2\,\mu_y}\,\Bigg(e^{\,2\,(\sigma_x^2+\sigma_y^2+2\rho_{x,y}\sigma_x\sigma_y)} - e^{\,\sigma_x^2+\sigma_y^2+2\rho_{x,y}\sigma_x\sigma_y}\Bigg)$$
$$= (100^2)(0.50^2)\,Exp\Bigg\{(2)(0.20)+(2)(0.05)\Bigg\}\Bigg[Exp\Bigg\{(2)(0.40^2+0.10^2+(2)(0.60)(0.40)(0.10))\Bigg\}$$
$$- Exp\Bigg\{0.40^2+0.10^2+(2)(0.60)(0.40)(0.10)\Bigg\}\Bigg]$$
$$= 1,248.58 \tag{47}$$